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There Is One Group of Genus Two

THOMAS W. TUCKER*

Colgate University, Hamilton, New York 13346

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The genus of a finite group is the minimum genus over all surfaces containing an imbedded Cayley graph for the group. It is shown that there is exactly one group of genus two.

The genus of a finite group is the minimum genus over all closed, orientable surfaces containing an imbedded Cayley graph for the group. White introduced the genus of a group in [13]; however as early as 1896, Mashke [6] classified all groups having a planar Cayley graph, that is all groups of genus zero. In 1931 Baker [1] listed groups having Cayley graphs with semi-regular imbeddings in the torus. In her thesis, Proulx [7] dramatically generalized Baker's work by classifying all groups of genus one according to thirty partial group presentations, most of which are satisfied by infinitely many groups. Proulx showed in her thesis that all but four of these presentations are quotients of the 17 2-dimensional Euclidean space groups (for the correspondence between the presentations as announced in [8] and the space groups as tabulated in [2], see [10]). The status of the four remaining presentations, each corresponding to a single group, was settled in [10], where it is shown that three of the four are not space group quotients; one of these exceptional groups plays an important role in this paper.

Also in her thesis, Proulx computed the genus of every group of order 31 or less except the abelian group $C_3 \times C_3 \times C_3$. She was led by this evidence and her computations for toroidal groups to conjecture that there are no groups of genus two. As a first step towards proving Proulx's conjecture, we showed [12] that there are only finitely many groups of a given genus greater than one. In fact, the main result of [12] is a surprisingly exact analogue to Hurwitz's theorem [5] bounding the order of a group of conformal automorphisms of a Riemann surface of given genus greater than one. Further connections between Cayley graph imbeddings and groups acting on surfaces, especially actions involving reflections, were uncovered in

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[11]. Finally, in [9] nearly all known results about groups of a given genus were summarized and extended in refined Hurwitz theorems that give detailed information about the structure of any group whose order is sufficiently large compared to its genus. Most of the work in settling Proulx's conjecture is really in [9]—this paper could be considered an appendix to [9].

THEOREM. *There is exactly one group C of genus two. Its order is* 96, *and it has the presentation*:

$$\langle x, y, z : x^2 = y^2 = z^2 = 1,\ (xy)^2 = (yz)^3 = (xz)^8 = 1, y(xz)^4 y(xz)^4 = 1\rangle.$$

We knew nothing about the group C when it first arose in our computations, but Coxeter subsequently informed us of its role in his paper [3]. He has since given a detailed description of the group C in [4], which is highly recommended for an account of the algebraic, analytic, and geometric origins of this group.

The theorem is proved in Sections 2, 3, and 4 by considering, respectively, candidate groups of order less than 48, equal to 48, and greater than 48. Section 1 recalls from [9] the necessary facts and terminology about triangle group quotients.

1. DEFINITIONS

As in [9], an ordered pair (x, y) of generators for a group is said to be $(p, q; r)^o$ if x has order p, y has order q, and either xy or xy^{-1} has order r. An ordered triple (x, y, z) of generators for a group is said to be (p, q, r) if x, y, and z all have order 2 and xy has order p, yz has order q, and xz has order r; such a triple is called *properly* (p, q, r) if the subgroup generated by xy and yz has index 2 (it is easily shown that it must have index at most 2). An ordered pair (x, y) of generators for a group is said to be $(p, q; q)^c$ if x has order p, y has order 2, and the commutator $xyx^{-1}y$ has order q; such a pair is *properly* $(p, p; q)^c$ if the subgroup generated by x and yxy has index 2 (again its index is necessarily at most 2). A group is called $(p, q; r)^o$, (p, q, r), etc., if it has a generating set that is $(p, q; r)^o$, etc.

These special types of generating sets are discussed at length in Section 2 of [9]. The important facts we need here are the following ($|A|$ denotes the order of A):

(1) a $(p, q; r)^o$ group A has a Cayley graph imbeddable in a surface of genus $1 + |A|(1 - 1/p - 1/q - 1/r)/2$;

(2) a proper (p, q, r) group A has a Cayley graph imbeddable in a surface of genus $1 + |A|(1 - 1/p - 1/q - 1/r)/4$;

(3) a proper $(p, p; q)^c$ group A has a Cayley graph imbeddable in a surface of genus $1 + |A|(1 - 2/p - 1/q)/4$.

The rest of our notation and terminology follows [9] as well. We let $\langle x, y\rangle$ denote the subgroup generated by x and y. *Relators* and *reduced relators* are defined as in [9]. A relator is *mixed* if it involves more than one generator. A generating set for a group is *redundant* if a proper subset generates the group; otherwise the generating set is *irredundant*. The cyclic group of order n is denoted C_n, the dihedral group of order $2n$ is denoted D_n, and the full symmetric group on n symbols is denoted S_n. Our group theoretic arguments are elementary. The one fact we use continually is that if the group A has a subgroup B of index n, then there is a transitive permutation representation of A in S_n; that is, there is a homomorphism $f: A \to S_n$ such that $f(A)$ is transitive on n symbols, and the kernel of f is contained in B.

2. Groups of Order Less Than 48

In her thesis [7], Proulx shows that every group of order less than 32 has genus at most 1 except for two groups and these two have genus at least 3. In particular, we need only consider groups A satisfying $|A| > 12|\chi| = 24$, and hence our refined Hurwitz theorems [9] apply. Inspection of these theorems shows we must study groups A of genus γ with irredundant generating sets of the following types:

(a) $(2, 3; r)^o$,

(b) $(5, 2; 4)^o$,

(c) $(2, 5, r)$, $r > 4$,

(d) $(5, 5; 2)^c$, proper,

(e) $(2, 4, r)$, $r > 4$,

(f) $(2, 3, r)$, $r > 6$,

(g) $(3, 3, r)$, $r = 4, 5$.

(a) For $(2, 3; r)^o$ groups, if there is a reduced mixed relator of length less than 14, then $\gamma \leqslant 1$ by the proof of Theorem 5.1 of [9]. It is simple to enumerate all words of length 6 or less beginning with x or x^{-1} and alternating y with $x^{\pm 1}$; there are 28. Thus there are also 28 such words of length 7 or less that begin with y. Since $|A| < 48$, there must be a reduced, mixed relator of length 13 or less by the pigeon hole principle.

(b, c, and d) In each of these cases, 10 divides $|A|$ and hence $|A| = 40$, since $31 < |A| < 48$ by assumption. By Sylow theory, the number of subgroups of order 5 divides $|A| = 40$ and is conjugate to 1 modulo 5; hence there is one subgroup of order 5 and it is normal. Thus, for cases (b) and (d), if x, y is $(5, 2; 4)^o$ or $(5, 5; 2)^c$, then the subgroup $\langle x \rangle$ is normal. Since y has order 2, this implies $|A| = 10$, a contradiction. For case (c), if x, y, z is $(2, 5, r)$, then $\langle yz \rangle$ is normal. Since $xyx = y$, it follows that $\langle y, z \rangle = \langle y, yz \rangle$ is normal. Because $|\langle y, z \rangle| = 10$ and x has order 2, this implies $|A| = 20$, a contradiction.

(e) Suppose x, y, z is a $(2, 4, r)$ generating set for the group A, $r > 4$. Then $\langle y, z \rangle$ and $\langle x, z \rangle$ are the dihedral groups D_4 and D_r, and hence 8 and $2r$ divide $|A|$. Since $31 < |A| < 48$, the only possibilities are $r = 5, 8, 10$. The case $r = 5$ is handlel by Sylow theory as in case (c). If $r = 8$ or 10, then $|A| = 32$ or 40, respectively. It follows in both cases that $\langle x, z \rangle$ has index 2 and is normal. Therefore $\langle xz \rangle$ is normal of index 4. Since A is generated by involutions, the quotient of A by $\langle xz \rangle$ must be Klein's 4-group, D_2. In particular, $(yz)^2 \in \langle xz \rangle$. But $\langle xz \rangle$ is normal in $\langle x, z \rangle$ and has only one element of order 2; therefore $x(yz)^2 x = (yz)^2$, since $(yz)^2$ has order 2. This relation can be simplified using $xy = yx$ to obtain the relator $(xzyz)^2$, which implies $\gamma \leqslant 1$ by presentation 3.4 of [8].

(f, g) By fact 2 in Section 1, the order of any proper $(2, 3, r)$ group is divisible by $24r/(r-6)$. Thus there is no such group with $r > 6$ and order less than 48. Any improper $(2, 3, r)$ group is $(2, 3; r)^o$ and such groups of order less than 48 have already been eliminated. Similarly, since the order of any $(3, 3, r)$ is divisible by 3 and $2r$, there is no such group with $r = 4, 5$ and order between 31 and 48.

3. Groups of Order 48

The analysis of $(2, 3; r)^o$ groups in the previous section actually showed that $\gamma \leqslant 1$ for such groups as long as their order is 56 or less. Since the order of any $(2, 3, r)$ group is divisible by $2r$, the only possible ones of order 48 with $r > 6$ are $(2, 3, 8)$ or $(2, 3, 12)$. A proper $(2, 3, 8)$ group has order divisible by 96 by fact 1; Section 1; thus it is improperly $(2, 3, 8)$, but then it is $(2, 3; 8)^o$ and hence $\gamma \leqslant 1$. If x, y, z is a $(2, 3, 12)$ generating set for a group A of order 48, then $\langle x, z \rangle$ is normal of index 2. Then $yzy \in \langle x, z \rangle$, but $yzy = zyz$ so x and z generate A, a contradiction. We conclude that the only possible generating sets are $(2, 4, r)$, $r > 4$, or $(3, 3, 4)$.

Possibilities for r are 6, 8, and 12. If $r = 12$, then $\langle x, z \rangle$ is normal of index two; it follows that $\gamma \leqslant 1$ by the same argument as case (e), $r = 8$ or 10, in the previous section. Thus we need only consider $r = 6$ or 8. If $r = 8$, then $\langle x, z \rangle$ has index 3, so there is a transitive representation $f: A \to S_3$. However, because $f(xy)$, $f(yz)$, and $f(xz)$ all have even order, the representation cannot be transitive.

Suppose instead that $r = 6$. Then there is a transitive representation $f: A \to S_4$ with kernel B contained in $\langle x, z \rangle$. We claim that $f(xy)$ has order 2,

$f(yz)$ has order 4, and $f(xz)$ has order 3. Certainly $f(xy)$ has order 2 since $xy \notin \langle x, z\rangle$. Similarly $f(yz)$ has order 2 or 4. If $(yz)^2 \in \langle x, z\rangle$ then $yzy \in \langle x, z\rangle$. However, this means $\langle x, z\rangle$ is normal, since $yxy = x$ already. If $\langle x, z\rangle$ is normal, it must have index 2, which contradicts the assumed order of A. Therefore $(yz)^2 \notin \langle x, z\rangle$ and $f(yz)$ has order 4. The order of $f(xz)$ cannot be 1 since $f(xz) = f(xy)f(yz)$, nor can it be 6 since no element of S_4 has order 6. Thus $f(xz)$ has order 2 or 3. If it has order 2, then $(xz)^2 \in B$. As B is properly contained in $\langle x, z\rangle$, it has order 3 or 6, so $(xz)^2$ and $(zx)^2$ are its only elements of order 3. Thus $y(xz)^2 y = (xz)^2$ or $(zx)^2$. If $y(xz)^2 y = (xz)^2$ then using $xyx = y$ we obtain the relator $yzxzyzxz$ which implies $\gamma \leqslant 1$ by 3.4 of [8]. If $y(xz)^2 y = (zx)^2$ then $xyzxzxyzxz$ is a relator. If we let $u = xy$ then A is generated by x, u, z and has relators u^2, $(xu)^2$, and $uzxzuzxz$ which again yields $\gamma \leqslant 1$. We conclude that $f(xy)$ has order 2, $f(yz)$ has order 4, and $f(xz)$ has order 3. Thus $f(A)$ is a (2, 4, 3) group. If it were properly (2, 4, 3) it would have order 48 which is impossible since $f(A) \subset S_4$. Therefore $f(A) = S_4$ and is an improper (2, 4, 3) group. A (p, q, r) group is improper if and only if it has a relator of odd length (see [9]). Since the kernel of f is $\langle (xz)^3\rangle$, $f(A)$ is obtained from A by adding the even relator $(xz)^3$. Therefore, as $f(A)$ is improperly (2, 4, 3), it follows that A is improperly (2, 4, 6). Thus by Proposition 2.1 of [9], the Cayley graph for A corresponding to x, y, z, does not have genus 2.

Let x, y, z be a (3, 3, 4) generating set for the group A. Since A has a subgroup of order 16, there is a transitive representation $f: A \to S_3$ with kernel B of order 8. Since no element of S_3 has order 4, $f(xz)$ has order at most 2, and hence $(xz)^2 \in B$. The orders of xy and yz prohibit them from being in B, and therefore they are not in the normal closure N of $(xz)^2$ either. Thus the quotient A/N is a (3, 3, 2) group. Since A/N then has order 24, the order of N is 2, that is, $(xz)^2$ is normal. Therefore $xzxz = yxzxzy$ so $zx = xyxzxzyz = yxyzxyzy$, and hence $(xyz)^2$ is conjugate to zx. Thus xyz has order 8. If we let $u = xy$, then u, y, z is a (2, 3, 8) generating set for A which implies by previous analysis that $\gamma \leqslant 1$.

4. Groups of Order Greater than 48

By [9], we need only consider a $(2, 3; 7)^o$ group of order 84, a proper (2, 4, 5) group of order 80, and proper $(2, 3, r)$ groups of order $24r/(r-6)$, $7 \leqslant r \leqslant 10$. Since we have already shown there is no $(2, 4; 5)^o$ group of order 40, the proper (2, 4, 5) group of order 80 is also impossible. It is well known that the smallest $(2, 3; 7)^o$ group is Klein's simple group of order 168; in any case, it follows from a Sylow theory argument like case (a) in Section 2 that there is no $(2, 3; 7)^o$ group of order 84. Therefore, there is no proper (2, 3, 7) of order 168 either. We will show that there is no (2, 3, 10) group of order

60 and no $(2, 3, 9)$ group of order 72. Finally, we show that there is exactly one proper $(2, 3, 8)$ group of order 96, namely the group of our theorem, and that this group does not have genus less than 2 (we know it has genus at most 2 by fact 2, Sect. 1).

Suppose x, y, z is a $(2, 3, 10)$ generating set for the group A of order 60. Then there is a transitive representation $f: A \to S_3$ with kernel B contained in the subgroup $\langle x, z\rangle$. Clearly $f(xz)$ cannot have order 10 or 5, so $(xz)^2 \in B$. Thus $\langle (xz)^2\rangle$ is normal because it contains the only elements of order 5 in the subgroup $\langle x, z\rangle$. Since the group C_5 has no automorphisms of order 3, yz must commute with $(xz)^2$. Thus $yz \cdot xzxz = xzxz \cdot yz = xzxyzy$. Therefore $yxzxzxzy = zxyz$ which implies $(xz)^3$ has order 2, contradicting the assumed order of xz.

Suppose x, y, z is a $(2, 3, 9)$ generating set for the group A of order 72. Then there is a transitive representation $f: A \to S_4$ with kernel B contained in the subgroup $\langle x, z\rangle$. Clearly $f(xz)$ cannot have order 9 so $(xz)^3 \in B$. Thus $\langle (xz)^3\rangle$ is normal because it contains the only elements of order 3 in the subgroup $\langle x, z\rangle$. It follows that yz commutes with $(xz)^3$ since C_3 has no automorphism of order 3. Thus $yz(xz)^3 = (xz)^3\, yz = (xz)^2\, xyzy$. Therefore $y(xz)^4\, y = zxzyxz$ which implies $(xz)^4$ has order 3 contradicting the assumed order of xz.

Finally, let x, y, z be a $(2, 3, 8)$ generating set for a group A of order 96. Then there is a transitive representation $f: A \to S_6$ with kernel B contained in the subgroup $\langle x, z\rangle$. Since $f(xz)$ cannot have order 8, $(xz)^4 \in B$. Thus $y(xz)^4\, y \in \langle x, z\rangle$. Since $(xz)^4$ is the unique central element of the dihedral group $\langle x, z\rangle$, it follows that $y(xz)^4\, y = (xz)^4$. Therefore every relator in the group C of our theorem is a relator in A, making A a quotient of C. But A and C both have order 96, so A is isomorphic to C.

It remains to show that the group C of our theorem does not have genus 1. Suppose it did. Then by Proulx's classification [8] of toroidal groups, C is a quotient of a 2-dimensional Euclidean space group (notice C cannot be one of the three exceptional toroidal groups [10] since its order is wrong). Therefore the subgroup $\langle xy, yz\rangle$ is also a space group quotient. However, if we let $u = xy$ and $v = yz$ then the relator $yz(xz)^4\, zy(xz)^4$ becomes the relator $v(uv)^4\, v^{-1}(uv)^4 = (vu)^4\, (uv)^4 = (vu)^3\, v^{-1}(uv)^3$, which yields the relator $(uvuvuv^{-1})^2$. The group $\langle u, v\colon u^2 = v^3 = (uvuvuv^{-1})^2 = 1\rangle$ is the group (2.8) of order 48 in Proulx's classification. In [10] we showed it is not the quotient of a 2-dimensional Euclidean space group. Therefore C does not have genus 1.

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